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Orthocomplementation and Compound Systems

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In their 1936 founding paper on quantum logic, Birkhoff and von Neumann postulated that the lattice describing the experimental propositions concerning a quantum system is orthocomplemented. We prove that this postulate fails for the lattice $\mathcal{L}_{\rm sep}$ describing a compound system consisting of so called *separated* quantum systems. By separated we mean two systems prepared in different "rooms" of the lab, and before any interaction takes place. In that case, the state of the compound system is necessarily a product state. As a consequence, Dirac's superposition principle fails, and therefore $\mathcal{L}_{\rm sep}$ cannot satisfy all Piron's axioms. In previous works, assuming that $\mathcal{L}_{\rm sep}$ is orthocomplemented, it was argued that $\mathcal{L}_{\rm sep}$ is not orthomodular and fails to have the covering property. Here we prove that $\mathcal{L}_{\rm sep}$ cannot admit an orthocomplementation. Moreover, we propose a natural model for $\mathcal{L}_{\rm sep}$ which has the covering property.

KEY WORDS: quantum logic; compound system; ortholattice; tensor product.

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1. INTRODUCTION

A cornerstone in physics is the concept of a mathematical phase-space Σ_S associated with a physical system S, representing all possible states of S. For instance, a classical particle is at each instant t associated with a point $(\vec{x}_t, \vec{p}_t) \in \mathbb{R}^6$ where $\vec{x}_t$ and $\vec{p}_t$ are the position and the momentum of the particle at time t, respectively. On the other hand, in quantum theory, it is assumed that there is a complex Hilbert space $\mathcal{H}_S$ associated with S, such that $\Sigma_S = (\mathcal{H}_S - 0)/\mathbb{C}$, the set of one-dimensional subspaces of $\mathcal{H}_S$ (Neumann, 1955).

In Birkhoff and Neumann (1936), Section 2, Birkhoff and von Neumann call a measurement $\mathcal{M}$ on a physical system S, together with a given subset σ of possible outcomes, an *experimental proposition* concerning the system S. Experimental propositions can be correlated with subsets of Σ_S by assigning to each proposition P, the set $\mu(P)$ of states in which the measurement yields with certainty an outcome in σ. In the sequel, we shall denote the image of the map μ,

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ordered by set-inclusion, by $\mathcal{L}_S$. Note that $\mathcal{L}_S \subseteq 2^{\Sigma_S}$, and obviously $\emptyset$ and Σ_S are in $\mathcal{L}_S$.

In classical mechanics, $\mu(\neg P) = \Sigma_S \backslash \mu(P)$, where $\neg P$ denotes the proposition defined by the same measurement, but with the complementary subset of outcomes $\mathcal{O}_{\mathcal{M}} \backslash \sigma$, where $\mathcal{O}_{\mathcal{M}}$ denotes the set of all possible outcomes of the measurement $\mathcal{M}$. This means that for any state, the probability that the outcome lies in σ (respectively in $\mathcal{O}_{\mathcal{M}} \backslash \sigma$) is either 1 or 0. Hence, in classical mechanics, $\mathcal{L}_S$ is a suborthoposet of 2^{Σ_S}.

For quantum theory the situation is totally different: The measurement $\mathcal{M}$ is associated with a self-adjoint operator in $\mathcal{H}_S$, and σ with a projector P_V on a closed subspace V of $\mathcal{H}_S$. For a given state p, the probability that the outcome of $\mathcal{M}$ lies in σ (respectively in $\mathcal{O}_{\mathcal{M}} \backslash \sigma$) is given by $\|\mathsf{P}_V(\phi)\|^2$ (respectively by $\|\mathsf{P}_{V^\perp}(\phi)\|^2$) where $\phi \in p$ with $\|\phi\| = 1$. Whence, $\mu(P) = (V - 0)/\mathbb{C}$ and $\mu(\neg P) = (V^\perp - 0)/\mathbb{C}$. Therefore, in quantum theory, $\mathcal{L}_S$ is a suborthoposet of $\mathsf{P}(\mathcal{H}_S) = \{(V - 0)/\mathbb{C}\,;\ V \subseteq \mathcal{H}_S,\ V^{\perp\perp} = V\}$, the lattice of closed subspaces of $\mathcal{H}_S$.

For both classical mechanics and quantum theory, Birkhoff and von Neumann postulated that $\mathcal{L}_S$ is an orthocomplemented lattice (Birkhoff and Neumann (1936), Section 5–6). More precisely, $\mathcal{L}_S$ is assumed to be a subortholattice, of 2^{Σ_S} in the classical case, and of $\mathsf{P}(\mathcal{H}_S)$ in the quantum case.

In this paper, we want to study the mathematical structure of the phase-space Σ_S and the poset $\mathcal{L}_S$ of a compound system S consisting of two *separated* quantum systems S_1 and S_2. By separated, we mean two systems (electrons, atoms or whatever) prepared in two different "rooms" of the lab, and before any interaction takes place. In that case, we denote Σ_S by Σ_{sep}, and $\mathcal{L}_S$ by $\mathcal{L}_{\text{sep}}$. As a main result, we show that $\mathcal{L}_{\text{sep}}$ cannot admit an orthocomplementation.

What do we know about Σ_{sep} and $\mathcal{L}_{\text{sep}}$? In quantum theory, the phase-space of a two-body system is given by $(\mathcal{H}_1 \otimes \mathcal{H}_2 - 0)/\mathbb{C}$, hence the state of S can be either entangled or a product state (Neumann, 1955). Entangled states have been observed in many experiments, involving pairs of photons (see Aspect (1999)) and references herein) or massive particles (Rowe *et al.*, 2001). Gisin proved that any entangled state violates a Bell inequality (Gisin, 1991). Therefore, for separated systems as defined here, the state is necessarily a product $p_1 \otimes p_2$ with $p_i \in (\mathcal{H}_i - 0)/\mathbb{C}$. Whether the two systems are fermions or bosons does not matter. Since they are prepared independently and do not interact, they are distinguishable and not correlated. As a consequence, we can put

$$\Sigma_{\text{sep}} = \Sigma_{S_1} \times \Sigma_{S_2}\,.$$

Further, let P_1 and P_2 be experimental propositions concerning S_1 and S_2, respectively. Then, obviously, both P_1 and P_2 are also experimental propositions concerning the compound system S. Moreover, $\mu(P_1) = \mu_1(P_1) \times \Sigma_2$ and $\mu(P_2) = \Sigma_1 \times \mu_2(P_2)$. Now, since S_1 and S_2 are totally independent from each other, we can perform P_1 and P_2 simultaneously (or one after the other) and define the experimental propositions concerning the compound system P_1 AND P_2 and P_1 OR P_2. Then, obviously

$$\mu(P_1 \text{ AND } P_2) = \mu_1(P_1) \times \mu_2(P_2)$$

$$\mu(P_1 \text{ OR } P_2) = \mu_1(P_1) \times \Sigma_2 \cup \Sigma_1 \times \mu_2(P_2)$$

Note that if we only consider those kind of experimental propositions on the compound system S, then $\mathcal{L}_{\text{sep}}$ is given by the *separated product* of Aerts $\mathcal{L}_{S_1} \textcircled{\wedge} \mathcal{L}_{S_2}$ defined in Aerts (1982) (see Section 4).

In Section 2, we will see that some important experimental propositions are not described by the separated product of Aerts. This means that $\mathcal{L}_{\text{sep}}$ cannot be constructed by simply considering the conjunctions and disjunctions of propositions concerning S_1 and S_2. As a consequence, in order to investigate the mathematical structure of $\mathcal{L}_{\text{sep}}$, we proceed as follows. First, we show that $\mathcal{L}_{\text{sep}}$ is a *weak tensor product* of $\mathcal{L}_{S_1}$ and $\mathcal{L}_{S_2}$ (Section 3), and then, we prove that if a weak tensor product admits an orthocomplementation, then it is isomorphic to the separated product of Aerts; whence follows our main claim, namely that $\mathcal{L}_{\text{sep}}$ cannot admit an orthocomplementation.

2. TWO ARGUMENTS AGAINST THE SEPARATED PRODUCT

2.1. Missing Propositions

Let S be any physical system undergoing some time evolution from a time t_0 to a time t_1. Let $U : \Sigma_{S t_0} \to \Sigma_{S t_1}$ be a map describing this time evolution. Let $\mathcal{M}_{t_1}$ be a measurement which can be performed on S at time t_1, and let P_{t_1} be an experimental proposition associated with $\mathcal{M}_{t_1}$. Then, Daniel pointed out that we can define an experimental proposition $\Phi(P_{t_1})$ concerning S at time t_0, by the prescription: "Let S evolve from time t_0 to time t_1 and perform $\mathcal{M}_{t_1}$"; obviously, $U^{-1}(\mu(P_{t_1})) = \mu(\Phi(P_{t_1}))$ (Daniel, 1989). As a consequence, if we ask $\mathcal{L}_{S t_0}$ to describe also those kind of experimental propositions, then for any $b \in \mathcal{L}_{S t_1}$, $U^{-1}(b) \in \mathcal{L}_{S t_0}$.

Note that in quantum theory, the time evolution of an isolated system S is described by a unitary operator on the Hilbert space $\mathcal{H}_S$; moreover unitary operators preserve closed subspaces. In general, it seems natural to require that the poset representing the experimental propositions concerning a physical system, describes all experimental propositions defined by Daniel's prescription, applied to any possible time evolution. Is it true for the separated product? To answer this question, we must first know what kind of time evolutions can undergo two initially separated quantum systems S_1 and S_2.

First, one can simply keep each system in its own "room," and let the systems evolve. Consider now the experimental situation represented schematically

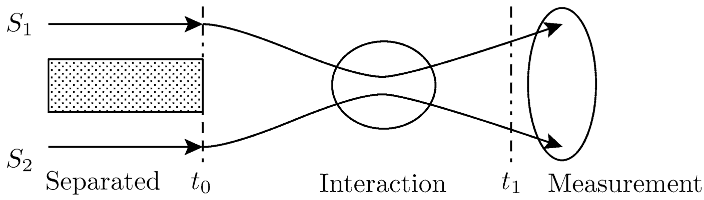

Fig. 1.

in Fig. 1. Two quantum systems S_1 and S_2 are prepared in two different "rooms" of the lab and stay in their own "room" until a time t_0. Then S_1 and S_2 interact, and finally a measurement is performed at some later time t_1, after the interaction has taken place. This is typically a situation encountered in scattering experiments.

According to quantum theory, the time evolution from t_0 to t_1 is given by a unitary operator $U = \exp[-i(t_1 - t_0)(H_1 \otimes 1 + 1 \otimes H_2 + W)]$, where H_i is the free Hamiltonian acting in $\mathcal{H}_i$, and W the interaction. As discussed in the introduction, at time t_0 the state of S is a product state. Now, because of the interaction W, U transforms instantaneously initial product states into entangled states. This means that the mathematical structure of the poset $\mathcal{L}_S$ of the compound system after time t_0 is very different from that of $\mathcal{L}_{\text{sep}}$. How this change occurs is far behind the scope of this article. However, we can expect that at time t_1, experimental propositions concerning the compound system correspond to closed subspaces of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Hence, according to Daniel's principle, it is natural to ask that for all $b \in \mathsf{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, the set of product states contained in $U^{-1}(b)$ is an element of $\mathcal{L}_{\text{sep}}$. This is not true if we put $\mathcal{L}_{\text{sep}} = \mathsf{P}(\mathcal{H}_1) \circledwedge \mathsf{P}(\mathcal{H}_2)$ (Ischi (submitted for publication), Theorem 10.4). As a consequence, some important experimental propositions are not described by the separated product of Aerts. Let us give a second argument against it.

2.2. Propensities

It is natural to assume the existence of a *propensity* map $\omega : \Sigma_{\text{sep}} \times \mathcal{L}_{\text{sep}} \to [0, 1]$ as defined for instance in Gisin (1984). By a result of Pool, every orthocomplemented lattice which admits a propensity map is orthomodular (Pool, 1968). Now, $\mathsf{P}(\mathcal{H}_1)$, $\mathsf{P}(\mathcal{H}_2)$ and $\mathsf{P}(\mathcal{H}_1) \circledwedge \mathsf{P}(\mathcal{H}_2)$ are complete atomistic orthocomplemented lattices. Hence, if $\mathcal{L}_{\text{sep}} = \mathsf{P}(\mathcal{H}_1) \circledwedge \mathsf{P}(\mathcal{H}_2)$, then $\mathcal{L}_{\text{sep}}$ cannot admit a propensity map, since by Aerts's theorem, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are complete atomistic orthocomplemented lattices and $\mathcal{L}_1 \circledwedge \mathcal{L}_2$ is orthomodular, then $\mathcal{L}_1$ or $\mathcal{L}_2$ is distributive (Aerts, 1982).

3. PHYSICAL HYPOTHESES

3.1. General Assumptions on the Posets Representing Experimental Propositions

In the sequel, for any physical system S, we shall consider not only experimental propositions, but more generally all $\{0, 1\}$-valued experiments on S. Hence, following Piron (1976) and Aerts (1982), we shall assume that $\mathcal{L}_S$ is closed under arbitrary set-intersections (*i.e.* $\cap\omega \in \mathcal{L}_S$, for all $\omega \subseteq \mathcal{L}_S$). Let us repeat the physical argument. Let $\{\mu(\alpha_i) \in \mathcal{L}_S\}_{i\in I}$ with α_i $\{0, 1\}$-valued experiments on S. Define $\pi_i\alpha_i$ by the prescription: "Perform any α_i." Then obviously, $\mu(\pi_i\alpha_i) = \cap_i \mu(\alpha_i)$.

We make a second general hypothesis on the posets representing experimental propositions. For $p \in \Sigma_S$, let ε_p denote all $\{0, 1\}$-valued experiments α on S, such that $p \in \mu(\alpha)$. Then, following Aerts (1982), we assume that for any two states p and q in Σ_S, $\varepsilon_p \not\subseteq \varepsilon_q$. Whence follows that $\{p\} = \cap\{\mu(\alpha)\,;\, \alpha \in \varepsilon_p\}$, for all $p \in \Sigma_S$.

As a consequence, $\mathcal{L}_S$ contains $\emptyset$, Σ_S, and all singletons of Σ_S, and $\mathcal{L}_S$ is closed under arbitrary set-intersections. Hence, $\mathcal{L}_S$ is the set of closed subspaces of a simple closure space.

To be short, we call a set $\mathcal{L}$ of subsets of a nonempty set Σ, closed under arbitrary set-intersections, and containing $\emptyset$, Σ, and all singletons of Σ, a *simple closure space* on Σ. Note that a simple closure space is a complete atomistic lattice. Moreover, if $\mathcal{L}$ is a complete atomistic lattice, then $\{\Sigma[a]\,;\, a \in \mathcal{L}\}$, where $\Sigma[a]$ denotes the set of atoms under a, is a simple closure space on the set of atoms of $\mathcal{L}$.

Four our main result, we need to assume moreover that $\mathcal{L}_{S_1}$ and $\mathcal{L}_{S_2}$ are orthocomplemented with the covering property, which is of course true if $\mathcal{L}_{S_i} = \mathsf{P}(\mathcal{H}_i)$ with $\mathcal{H}_i$ a complex Hilbert space.

3.2. Assumptions Relating $\mathcal{L}_{S_i}$ and $\mathcal{L}_{S_{\text{sep}}}$

We assume that $\mathcal{L}_{\text{sep}}$ is a *weak tensor product* of $\mathcal{L}_{S_1}$ and $\mathcal{L}_{S_2}$:

Definition 3.1. Let $\mathcal{L}_1 \subseteq 2^{\Sigma_1}$ and $\mathcal{L}_2 \subseteq 2^{\Sigma_2}$ be simple closure spaces on Σ_1 and Σ_2, respectively. Then, $\mathsf{S}(\mathcal{L}_1, \mathcal{L}_2)$ is defined as the set of all simple closure spaces $\mathcal{L} \subseteq 2^{\Sigma}$ on Σ such that

- P1 $\Sigma = \Sigma_1 \times \Sigma_2$,
- P2 $a_1 \times \Sigma_2 \cup \Sigma_1 \times a_2 \in \mathcal{L}, \forall a_1 \in \mathcal{L}_1, a_2 \in \mathcal{L}_2$,
- P3 $\forall p_i \in \Sigma_i,\ A_i \subseteq \Sigma_i,\ [p_1 \times A_2 \in \mathcal{L} \Rightarrow A_2 \in \mathcal{L}_2]$ and $[A_1 \times p_2 \in \mathcal{L} \Rightarrow A_1 \in \mathcal{L}_1]$.

We call elements of $\mathsf{S}(\mathcal{L}_1, \mathcal{L}_2)$ weak tensor products of $\mathcal{L}_1$ and $\mathcal{L}_2$. Let $T_i \subseteq \mathsf{Aut}(\mathcal{L}_i)$, where $\mathsf{Aut}(\mathcal{L}_i)$ denotes the group of automorphisms of $\mathcal{L}_i$ (*i.e.* bijective maps preserving all meets and joins). Then, we define $\mathcal{S}_{T_1T_2}(\mathcal{L}_1, \mathcal{L}_2)$ as the subset of all $\mathcal{L} \in \mathsf{S}(\mathcal{L}_1, \mathcal{L}_2)$ such that

$$\text{P4} \;\; \forall v_i \in T_i, \exists u \in \mathrm{Aut}(\mathcal{L}) \mid u(p_1, p_2) = (v_1(p_1), v_2(p_2)), \forall (p_1, p_2) \in \Sigma.$$

Note that for a simple closure space $\mathcal{L} \subseteq 2^{\Sigma}$ on Σ, we omit the brackets when writing singletons and call elements of Σ atoms. Moreover, for $u \in \mathrm{Aut}(\mathcal{L})$, we also write u for the bijective map on Σ induced by u.

For our main result, we only need Axioms P1–P3. Axioms P1 and P2 have already been discussed in the introduction. We now discuss Axioms P3 and P4.

Axiom P4: If S_1 and S_2 are quantum systems described by two complex Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, then it is indeed natural to assume that Axiom P4 holds for $(T_1, T_2) = (\mathsf{U}(\mathcal{H}_1), \mathsf{U}(\mathcal{H}_i))$, where $\mathsf{U}(\mathcal{H}_i)$ denotes the group of automorphisms of $\mathsf{P}(\mathcal{H}_i)$ induced by unitary maps. In words, it is natural to assume that products of unitary maps represent physical symmetries of the compound system. Of course, we can expect that Axiom P4 also holds for pairs of antiunitary maps. Suppose now that the automorphisms in T_1 and T_2 describe possible time evolutions of each system. Then, according to the discussion in Section 2.1, Axiom P4 must hold for T_1 and T_2.

Axiom P3: From the experimental standpoint, the system S_1 can certainly not be prepared in any given state. However, we can reasonably assume that there is at least one state (say p_0) in which S_1 can be prepared, whatever the system S_1 might be. Now, suppose that $p_0 \times B \in \mathcal{L}_{\text{sep}}$ and let $P \in \mathcal{P}_{\text{sep}}$ such that $\mu(P) = p_0 \times B$. Define a $\{0, 1\}-$valued experiment P_2 as: "Prepare system S_1 in room 1 in the state p_0 and perform P." Then obviously, P_2 is a $\{0, 1\}-$valued experiment on S_2, and $\mu_2(P_2) = B$, hence $B \in \mathcal{L}_2$. Therefore, Axiom P3 follows from Axiom P4, if T_1 and T_2 act transitively on Σ_{S_1} and Σ_{S_2}, respectively, and contain the identity, which is of course true if $T_1 = \mathsf{U}(\mathcal{H}_1)$ and $T_2 = \mathsf{U}(\mathcal{H}_2)$. It is important to note, that to justify Axiom P3, we need to assume both the existence for each system of a particular state in which each it can be prepared, and enough physical symmetries. Indeed, if for instance T_1 corresponds to automorphisms describing possible time evolutions of S_1, then T_1 certainly does not act transitively on Σ_{S_1}. Finally, not that if the system S_1 can be prepared in a given state p_0, and if U_1 is a time evolution sending the initial state p_0 to a final state p_1, then this does not mean that S_1 can be prepared in the state p_1.

4. MATHEMATICAL RESULTS

4.1. Generalities

Before we present some mathematical results concerning weak tensor products, we want to emphasize on the fact that obviously, a weak tensor product

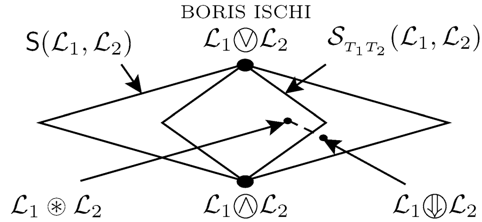

Fig. 2. T_i =Aut($\mathcal{L}_i$).

(hence $\mathcal{L}_{\text{sep}}$) cannot be isomorphic to the lattice of closed subspaces of a Hilbert space. Therefore, some of Piron's axioms must fail in $\mathcal{L}_{\text{sep}}$ (Piron, 1964). In the light of Theorem 34.5 in Maeda and Maeda (1970), one can expect that $\mathcal{L}_{\text{sep}}$ is not orthocomplemented with the covering property.

Definition 4.1. Let $\mathcal{L}_1 \subseteq 2^{\Sigma_1}$ and $\mathcal{L}_2 \subseteq 2^{\Sigma_2}$ be simple closure spaces on Σ_1 and Σ_2, respectively. Then

$$\mathcal{L}_1 \textcircled{\wedge} \mathcal{L}_2 := \{\cap\omega\,;\, \omega \subseteq \{a_1 \times \Sigma_2 \cup \Sigma_1 \times a_2\,;\, a_1 \in \mathcal{L}_1,\ a_2 \in \mathcal{L}_2\}\}\,,$$

$$\mathcal{L}_1 \textcircled{\vee} \mathcal{L}_2 := \{R \subseteq \Sigma_1 \times \Sigma_2\,;\, R_1[p] \in \mathcal{L}_1,\ R_2[p] \in \mathcal{L}_2,\ \forall p \in \Sigma_1 \times \Sigma_2\}\,,$$

ordered by set-inclusion, where $R_1[(p_1, p_2)] := \{s \in \Sigma_1\,;\, (s, p_2) \in R\}$ and similarly, $R_2[(p_1, p_2)] := \{t \in \Sigma_2\,;\, (p_1, t) \in R\}$.

As a first result, we find easily that for any $T_i \subseteq \mathsf{Aut}(\mathcal{L}_i)$, $\mathcal{S}_{T_1T_2}(\mathcal{L}_1, \mathcal{L}_2)$, ordered by set-inclusion, is a complete lattice, the bottom and top elements of which are given by $\mathcal{L}_1 \textcircled{\wedge} \mathcal{L}_2$ and $\mathcal{L}_1 \textcircled{\vee} \mathcal{L}_2$, respectively (Ischi (submitted for publication), Theorem 2.13). The meet is the set-intersection, hence $\mathcal{S}_{T_1T_2}(\mathcal{L}_1, \mathcal{L}_2)$ is a meet-sublattice of $\mathsf{S}(\mathcal{L}_1, \mathcal{L}_2)$.

Moreover, suppose that if $\mathcal{L}_i \neq 2^{\Sigma_i}$, then there are two atoms, say p and q, such that $p \vee q$ contains a third atom (say r) and covers p, q and r. Then, $\mathcal{L}_1 \textcircled{\wedge} \mathcal{L}_2 = \mathcal{L}_1 \textcircled{\vee} \mathcal{L}_2$ if and only if $\mathcal{L}_1 = 2^{\Sigma_1}$ or $\mathcal{L}_2 = 2^{\Sigma_2}$ (Ischi (submitted for publication), Theorems 5.2 and 5.4).

The bottom element $\mathcal{L}_1 \textcircled{\wedge} \mathcal{L}_2$ is the *separated product* of Aerts defined for ortholattices in Aerts (1982) (Ischi (submitted for publication), Lemma 3.2). For atomistic lattices (not complete) with 1, $\mathcal{L}_1 \textcircled{\wedge} \mathcal{L}_2$ can be defined in a similar way by taking only finite intersections. Then, it is isomorphic to the *box product* $\mathcal{L}_1 \Box \mathcal{L}_2$ of Grätzer and Wehrung (1999), and if $\mathcal{L}_1$ and $\mathcal{L}_2$ are moreover coatomistic, to the *lattice tensor product* $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ (Ischi (submitted for publication), Theorem 3.8).

On the other hand, the top element $\mathcal{L}_1 \textcircled{\vee} \mathcal{L}_2$ is the $\boxtimes-$tensor product of Golfin (1987), and it is isomorphic to the tensor products of Chu, Barr (1979) and Shmuely (1974) (Ischi (submitted for publication), Theorem 3.14). Let **C** be the category of complete join-semilattices with maps preserving arbitrary joins, and **c** the subcategory of **C** defined by considering as objects simple closure spaces. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be simple closure spaces. Then there is a bimorphism $f : \mathcal{L}_1 \times \mathcal{L}_2 \to \mathcal{L}_1 \ovee \mathcal{L}_2$, such that for any object $\mathcal{L}$ of **C** or **c** and any bimorphism $g : \mathcal{L}_1 \times \mathcal{L}_2 \to \mathcal{L}$, there is a unique arrow h such that the diagram commutes

$$\begin{array}{ccc} \mathcal{L}_1 \times \mathcal{L}_2 & \xrightarrow{f} & \mathcal{L}_1 \ovee \mathcal{L}_2 \\ {\scriptstyle g}\downarrow & \swarrow {\scriptstyle !h} & \\ \mathcal{L} & & \end{array}$$

(Ischi, Theorem 3.20). For join-semilattices and maps preserving finite joins, this is exactly the definition of the *join-semilattice tensor product* given by Fraser in Fraser (1976). Hence, we can call the top element the *complete join-semilattice tensor product* or simply the tensor product in the category **c**.

Note that for any $T_i \subseteq \mathsf{Aut}(\mathcal{L}_i)$, $\mathcal{S}_{T_1T_2}(\mathcal{L}_1, \mathcal{L}_2)$ can be defined as the set of all simple closure spaces satisfying the above universal property with respect not to all objects $\mathcal{L}$ and bimorphisms g, but with respect to a given class of objects and bimorphisms (Ischi (submitted for publication), Theorem 4.4). Therefore, it is natural to call elements of $\mathsf{S}(\mathcal{L}_1, \mathcal{L}_2)$ weak tensor products of $\mathcal{L}_1$ and $\mathcal{L}_2$.

4.2. Orthocomplemented Weak Tensor Products

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are orthocomplemented simple closure spaces, then the binary relation on $\Sigma_1 \times \Sigma_2$, defined as $(p_1, p_2)\#(q_1, q_2) \Leftrightarrow p_1 \perp q_1$ or $p_2 \perp q_2$, induces an orthocomplementation of the separated product $\mathcal{L}_1 \owedge \mathcal{L}_2$. Coatoms have the form $(p_1, p_2)^\# = p_1^\perp \times \Sigma_2 \cup \Sigma_1 \times p_2^\perp$. Our main result states that the separated product is the only orthocomplemented weak tensor product. More precisely, we have:

Theorem 4.2. (Ischi (submitted for publication), Theorem 8.6) *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be orthocomplemented simple closure spaces with the covering property, and let $\mathcal{L} \in \mathsf{S}(\mathcal{L}_1, \mathcal{L}_2)$. Then $\mathcal{L}$ admits an orthocomplementation if and only if $\mathcal{L} = \mathcal{L}_1 \owedge \mathcal{L}_2$.*

We outline the proof in case $\mathcal{L}_i$ are irreducible and $\mathcal{L}$ is transitive, *i.e.* the action of $\mathsf{Aut}(\mathcal{L})$ on the set of atoms of $\mathcal{L}$ is transitive (note that this is a consequence of Axiom P4 if T_1 and T_2 act transitively on Σ_1 and Σ_2, respectively). First it follows easily from Axiom P3 that if x_1 is a coatom of $\mathcal{L}_1$ and x_2 is a coatom of $\mathcal{L}_2$, then $X := x_1 \times \Sigma_2 \cup \Sigma_1 \times x_2$ is a coatom of $\mathcal{L}$. We prove that all coatoms of $\mathcal{L}$ are of this form. Denote by $' : \mathcal{L} \to \mathcal{L}$ the orthocomplementation of $\mathcal{L}$. Then X' is an atom of $\mathcal{L}$, say p. Let q be another atom. Since $\mathcal{L}$ is transitive, there is an automorphism $u \in \mathsf{Aut}(\mathcal{L})$ such that $u(p) = q$. Define $u' : \mathcal{L} \to \mathcal{L}$ as $u'(a) := (u(a'))'$. Then u' is an automorphism of $\mathcal{L}$. Moreover, $q' = u(p)' = u(p'')' = u'(X)$. Then the proof follows directly from

Theorem 4.3. (Ischi (submitted for publication), Theorem 7.5) *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be simple closure spaces such that the join of any two atoms contains a third atom. Let $\mathcal{L} \in \mathsf{S}(\mathcal{L}_1, \mathcal{L}_2)$, and let $u \in \mathsf{Aut}(\mathcal{L})$. Then there is a permutation σ and two isomorphisms $v_i : \mathcal{L}_i \to \mathcal{L}_{\sigma(i)}$ $(i = 1, 2)$ such that for all $p \in \Sigma_1 \times \Sigma_2$, $u(p)_{\sigma(i)} = v_i(p_i)$.*

The proof relies on the following remarks: Let $p, q \in \Sigma_1 \times \Sigma_2$. (i) By Axiom P2, if $p_1 \neq q_1$ and $p_2 \neq q_2$, then $p \vee q$ does not contain a third atom. (ii) By Axiom P3, if $p_1 = q_1$, then $p \vee q = p_1 \times (p_2 \vee q_2)$, and the same kind of equality holds for left lateral joins of atoms. As a consequence, since u preserves joins, $u(p_1 \times \Sigma_2)$ is either of the form $q_1 \times \Sigma_2$ or of the form $\Sigma_1 \times q_2$, with q_i atoms.

A similar result to Theorem 4.2 was obtained in (Ischi (to appear)) for $\mathcal{L}_i = \mathsf{P}(\mathcal{H}_i)$ and with a set of axioms weaker than those used here and in previous works (Aerts and Daubechies, 1978; Pulmannova, 1985; Watanabe, 2003).

4.3. Weak Tensor Products with the Covering Property

It was proved by Aerts, in case $\mathcal{L}_1$ and $\mathcal{L}_2$ are orthocomplemented simple closure spaces, that if $\mathcal{L}_1 \owedge \mathcal{L}_2$ has the covering property or is orthomodular, then $\mathcal{L}_1 = 2^{\Sigma_1}$ or $\mathcal{L}_2 = 2^{\Sigma_2}$ (Aerts (1982), or see Ischi, Theorem 9.1).

The same result holds for the top element $\mathcal{L}_1 \ovee \mathcal{L}_2$. More precisely, assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ have the covering property and that if $\mathcal{L}_i \neq 2^{\Sigma_i}$, then there are four atoms p, q, r and s such that $p \vee q$ covers p, q, r and s. Then, $\mathcal{L}_1 \ovee \mathcal{L}_2$ has the covering property if and only if $\mathcal{L}_1 = 2^{\Sigma_1}$ or $\mathcal{L}_2 = 2^{\Sigma_2}$ (Ischi (submitted for publication), Theorem 9.4).

We now give an example of a weak tensor product with the covering property, which, as discussed in Section 2.1, is a very natural model for $\mathcal{L}_{\text{sep}}$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be complex Hilbert spaces and let $\mathcal{L}_1 = \mathsf{P}(\mathcal{H}_1)$ and $\mathcal{L}_2 = \mathsf{P}(\mathcal{H}_2)$ be the lattices of closed subspaces. Let V be a closed subspace of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Denoted by $\Sigma_{\Downarrow}[V]$, the set of atoms of $\mathsf{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ contained in V and spanned by product vectors. Define

$$\mathcal{L}_1 \textcircled{$\Downarrow$} \mathcal{L}_2 := \{\Sigma_{\Downarrow}[V] \mid V \in \mathsf{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)\}\,.$$

Then $\mathcal{L}_1 \textcircled{$\Downarrow$} \mathcal{L}_2 \in \mathcal{S}_{T_1 T_2}(\mathcal{L}_1, \mathcal{L}_2)$ with $(T_1, T_2) = (\mathsf{U}(\mathcal{H}_1), \mathsf{U}(\mathcal{H}_2))$, where $\mathsf{U}(\mathcal{H}_i)$ denotes the group of automorphisms of $\mathsf{P}(\mathcal{H}_i)$ induced by unitary maps. Note that the same inclusion holds for pairs of antiunitary maps, but not for $(T_1, T_2) = (\mathsf{Aut}(\mathcal{L}_1), \mathsf{Aut}(\mathcal{L}_2))$. Moreover, $\mathcal{L}_1 \textcircled{$\Downarrow$} \mathcal{L}_2$ is different from the top and the bottom elements, $\mathcal{L}_1 \textcircled{$\Downarrow$} \mathcal{L}_2$ is coatomistic and has the covering property (Ischi (submitted for publication), Theorem 10.4). As an example, consider the case where $\mathcal{H}_1$ and

$\mathcal{H}_2$ have finite dimensions. Then, there is a bijection between anti-linear maps from $\mathcal{H}_1$ to $\mathcal{H}_2$ and coatoms of $\mathcal{L}_1 ⨀ \mathcal{L}_2$, namely $A \mapsto \{p \times (A(p))^\perp \mid p \in \Sigma_1\}$ (Ischi (submitted for publication), Proposition 10.2).

4.4. A Second Example

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be coatomistic simple closure spaces. Define $\mathcal{L}_1 \circledast \mathcal{L}_2 := \{\cap\omega \mid \omega \subseteq \Sigma'_{\circledast}\}$, with

$$\Sigma'_{\circledast} := \{R \subset_{\neq} \Sigma_1 \times \Sigma_2 \mid R_1[p] \in \Sigma'_1 \cup \{\Sigma_1\} \text{ and } R_2[p] \in \Sigma'_2 \cup \{\Sigma_2\},\ \forall p \in \Sigma_1 \times \Sigma_2\},$$

where Σ'_i denotes the set of coatoms of $\mathcal{L}_i$ (hence $R_i[p]$ is either a coatom or Σ_i). Then $\mathcal{L}_1 \circledast \mathcal{L}_2 \in \mathcal{S}_{T_1T_2}(\mathcal{L}_1, \mathcal{L}_2)$ with $T_i = \mathsf{Aut}(\mathcal{L}_i)$ (Ischi and Seal (to appear), Theorem 7.8).

Let $\mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$ be the category of coatomistic simple closure spaces such that for any two coatoms x and y, and any two atoms p and q, there is an atom r and a coatom z with $r \notin x \cup y$ and $p, q \notin z$, with maps preserving arbitrary joins, sending atoms to atoms or 0, and with right adjoint sending coatoms to coatoms or 1. Then, $\mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}$ equipped with the bifunctor $\circledast$ and the functor ^{op} which sends a lattice to its dual, is $*-$autonomous (Ischi and Seal (to appear), Theorem 5.5), hence a model for Girard's linear logic (Barr, 1991).

Note also that there is a bijection between $\mathbf{Cal}^{\mathbf{0}}_{\mathbf{Sym}}(\mathcal{L}_1, \mathcal{L}_2^{op})$ and $\Sigma'_{\circledast} \cup \{1\}$, namely $f \mapsto \{p \times f(p) \mid p \in \Sigma_1\}$. Hence, for finite-dimensional Hilbert spaces, we have

$$\mathsf{P}(\mathcal{H}_1) ⨀ \mathsf{P}(\mathcal{H}_2) \subseteq \mathsf{P}(\mathcal{H}_1) \circledast \mathsf{P}(\mathcal{H}_2).$$

Therefore, according to the discussion of Section 2.1, $\mathsf{P}(\mathcal{H}_1) \circledast \mathsf{P}(\mathcal{H}_2)$ might, as well as $\mathsf{P}(\mathcal{H}_1) ⨀ \mathsf{P}(\mathcal{H}_2)$, be a good candidate for $\mathcal{L}_{\text{sep}}$.

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